Malaysian Journal of Mathematical Sciences 17(3): 363–378 (2023)
https://doi.org/10.47836/mjms.17.3.08

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Circular q-Rung Orthopair Fuzzy Set and Its Algebraic Properties

Yusoff, B.*[1,2], Kilicman, A.[2,3], Pratama, D.[1], and Hasni, R.[1]

[1] *Special Interest Group on Modelling and Data Analytics, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, Malaysia*
[2] *Laboratory of Cryptography, Analysis and Structure, Institute for Mathematical Research, Universiti Putra Malaysia, Malaysia*
[3] *Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, Malaysia*

E-mail: binyamin@umt.edu.my
**Corresponding author*

Received: 18 May 2023
Accepted: 20 July 2023

Abstract

Circular intuitionistic fuzzy sets (CIFS) are a recent extension of intuitionistic fuzzy sets (IFS) that can handle imprecise membership values effectively. However, its representation is limited to the space under the intuitionistic fuzzy interpretation triangle (IFIT). To address this, a new generalization of CIFS called circular q-rung orthopair fuzzy sets (Cq-ROFS) is proposed, extending the IFIT to cover a larger space of imprecision. Several relations and operations, including algebraic operations for Cq-ROFS are proposed. In addition, modal operators and their properties are then investigated.

Keywords: circular intuitionistic fuzzy sets; q-rung orthopair fuzzy sets; circular q-rung orthopair fuzzy sets; algebraic properties; modal operators.

1 Introduction

An intuitionistic fuzzy set (IFS) is a set developed to handle problems related to imprecise and incomplete information [7]. This set was introduced by Atanassov, which is a generalization of the fuzzy set (FS) theory [30]. In FS, an element is marked by the presence of its membership ($\mathcal{M}$) degree or value (i.e., the non-membership ($\mathcal{N}$) degree is directly complemented to it). Meanwhile, in IFS, it is indicated by the presence of its $\mathcal{M}$ and $\mathcal{N}$ degrees, where the sum of the two can be less than one (i.e., any hesitancy or incomplete information is allowed). This makes IFS more flexible and covers more uncertain events in the decision-making process. Several studies have been conducted to expand the IFS, including in aggregation operators [26], and correlation coefficient [17], to mention a few. In addition, many authors have applied the IFS to decision-making problems [1, 13].

IFS has experienced numerous developments, especially in terms of the relationship between $\mathcal{M}$ and $\mathcal{N}$ degrees. Initially, the IFS met the condition $\mathcal{M}+\mathcal{N} \leq 1$. However, to cater for the issue beyond this inequality (i.e., $\mathcal{M}+\mathcal{N} > 1$), Yager [28] then defined the Pythagorean fuzzy sets (PFS), which changed the constraining relation to $\mathcal{M}^2+\mathcal{N}^2 \leq 1$. Prior to that, Atanassov [8] proposed IFS of second type to deal with the same issue. In 2011, Ciucci [15] introduced the term orthopair as an alternative pair of $\mathcal{M}$ and $\mathcal{N}$ degrees. This gives rise to the generalized orthopair fuzzy sets or called q-rung orthopair fuzzy sets (q-ROFS), which satisfy $\mathcal{M}^q+\mathcal{N}^q \leq 1$ for any q positive integers [29]. Vassilev et al. [25] defined a similar concept called IFS of q-type to generalize the IFS. Note that this set can be reduced to IFS for $q = 1$, PFS for $q = 2$ and Fermatean fuzzy sets (FFS), which is another special form of q-ROFS with $q = 3$ [24]. Similarly, several studies have explored the q-ROFS in the cases of aggregation operations [21, 2], similarity measures [16, 5], and some applications in decision-making problems [4, 3]. In general, the expression of q-ROFS is acknowledged to provide greater flexibility and expressive power for decision-makers in representing their preferences compared to IFS [29].

In 1989, IFS was expanded from what was originally a singular point into an area in an intuitionistic fuzzy interpretation triangle (IFIT) with a rectangular shape called interval-valued IFS (IVIFS) [10]. The main motivation for this extension was to deal with imprecise of $\mathcal{M}$ and of $\mathcal{N}$ values. Recently, Atanassov introduced another extension of $\mathcal{M}$ and $\mathcal{N}$ interpretation into a circle called circular IFS (CIFS) [9]. This set is characterized by a 3-tuple containing $\mathcal{M}$, $\mathcal{N}$, and radius for each element. The difference with IFS lies in the existence of a circular imprecision area with radius r. Compared to IVIFS, CIFS has an equidistant center point and boundary, which is not necessarily true for IVIFS, as their boundaries can take various shapes and distances from the center point. The CIFS theory is still at an early stage of its development. Hence, not much research has been conducted on it. Initially, Atanassov [9] defined the basic relations and operations for CIFS with $r \in [0, 1]$, but then has been expanded to $r \in [0, \sqrt{2}]$ to cover the whole region in the IFIT [11]. Some studies on CIFS have been conducted, including distance measures [11, 14] and divergence measures for CIFS [20]. Other than that, some extensions of decision-making models under the CIFS environment have also been proposed recently, such as in technique for order preference by similarity to ideal solution (TOPSIS) [18, 6], multiple criteria optimization and compromise solution (VIKOR) [19], the integration of analytic hierarchy process (AHP) and VIKOR [23] and a general multiple criteria decision making (MCDM) model [12]. Table 2 (see Appendix) provides explanations for the abbreviations used in this article.

Although CIFS provides a better representation for modeling imprecise $\mathcal{M}$ and $\mathcal{N}$ degrees, it is limited to the space of IFIT, where the sum of $\mathcal{M}$ and $\mathcal{N}$ degrees is bounded by one. This limitation can be overcome by expanding the existing IFIT area in CIFS to include the larger space provided by q-ROFS. Therefore, this paper aims to define a Circular q-Rung Orthopair Fuzzy Set

(Cq-ROFS), which not only can model the imprecise $\mathcal{M}$ and $\mathcal{N}$ degrees but also cover a larger space of imprecision, such that $\mathcal{M}+\mathcal{N} > 1$ is allowed. We begin by systematically defining the set form of Cq-ROFS, along with the applicable relations and operations. The following discussion is focused on algebraic operations, including intersection, union, algebraic sum, and algebraic product. We also examine several properties, such as idempotency, inclusion, and absorption, to determine the behavior of these operations in Cq-ROFS. Additionally, we define some modal operators to enrich the theory of Cq-ROFS.

This paper is structured as follows: Section 2 provides some preliminaries, including IFS, q-ROFS, and CIFS. In Section 3, the proposed Cq-ROFS is defined. In Section 4, some modal operators over Cq-ROFS are introduced. Finally, the conclusion and future studies are provided in Section 5.

2 Preliminaries

Several foundational ideas are presented in this section, including IFS, q-ROFS, and CIFS for any non-empty set X.

Definition 2.1. [7] *An intuitionistic fuzzy set $\mathcal{A}$ (denoted IFS $\mathcal{A}$) in X is defined as an object with characteristics $\mathcal{A} = \{\langle x, \mathcal{M}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x)\rangle | x \in X\}$, where $\mathcal{M}_{\mathcal{A}} : X \to [0,1]$ as the degree of membership and $\mathcal{N}_{\mathcal{A}} : X \to [0,1]$ as the degree of non-membership that satisfy $0 \leq \mathcal{M}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{A}}(x) \leq 1$ for each $x \in X$. The value of $\mathcal{H}(x) = 1 - \mathcal{M}_{\mathcal{A}}(x) - \mathcal{N}_{\mathcal{A}}(x)$ represents how uncertain (or non-deterministic) the element $x \in X$ to the IFS $\mathcal{A}$. IFS(X) represents the collection of all IFSs.*

Clearly, IFS can be reduced to FS when $\mathcal{M}_{\mathcal{A}}(x)+\mathcal{N}_{\mathcal{A}}(x) = 1$ or $\mathcal{H}(x) = 0$ for every $x \in X$. Since then, several extensions of IFS have been proposed by modifying the constraining relationship between $\mathcal{M}$ and $\mathcal{N}$ degrees. One such extension is the q-ROFS, which is a generalization of IFS.

Definition 2.2. [29] *A q-rung ortopair fuzzy set $\mathcal{A}^*$ (denoted q-ROFS $\mathcal{A}^*$) in X is defined as an object with characteristics $\mathcal{A}^* = \{\langle \mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x)\rangle | x \in X\}$, where the function $\mathcal{M}_{\mathcal{A}^*} : X \to [0,1]$ indicates support for membership degree and $\mathcal{N}_{\mathcal{A}^*} : X \to [0,1]$ indicates support againts membership degree of $x \in X$ which satisfy $0 \leq \mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) \leq 1$ with integer $q \geq 1$.*

Note that, q-ROFS can be reduced to IFS for $q = 1$, PFS for $q = 2$, and FFS for $q = 3$ (see Figure 1). The sum of $\mathcal{M}$ and $\mathcal{N}$ values represents the distinction between them. Specifically, IFS has the form $\mathcal{M}+\mathcal{N} \leq 1$, PFS has the form $\mathcal{M}^2+\mathcal{N}^2 \leq 1$ and FFS has the form of $\mathcal{M}^3+\mathcal{N}^3 \leq 1$. By increasing the value of q, the range of acceptable orthopair space widens which allows decision-makers to express their beliefs about membership degrees with greater flexibility. These formulations align with Yager's [29] description of q-ROFS, which connects the $\mathcal{M}$ and $\mathcal{N}$ degrees at the q-th term level (for integer $q \geq 1$). It is obvious that the definitions provided above pertain exclusively the scenarios where the degrees of $\mathcal{M}$ and $\mathcal{N}$ are precise. Atanassov [9] suggests a further extension of IFS to capture the case where these degrees are imprecise. Under this extension, the $\mathcal{M}$ and $\mathcal{N}$ degrees, which were initially single-value coordinates, are represented as circular areas called CIFS.

Definition 2.3. [9] *A circular intuitionistic fuzzy set $\mathcal{A}_r$ (denoted CIFS $\mathcal{A}_r$) in X is defined as $\mathcal{A}_r = \{\langle x, \mathcal{M}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x); r\rangle | x \in X\}$, where $\mathcal{M}_{\mathcal{A}} : X \to [0,1]$ and $\mathcal{N}_{\mathcal{A}} : X \to [0,1]$ are membership function and non-membership function, respectively such that $0 \leq \mathcal{M}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{A}}(x) \leq 1$ and $r \in [0, \sqrt{2}]$ is a radius of the circle around each element $x \in X$. The collection of all CIFSs is expressed by CIFS(X).*

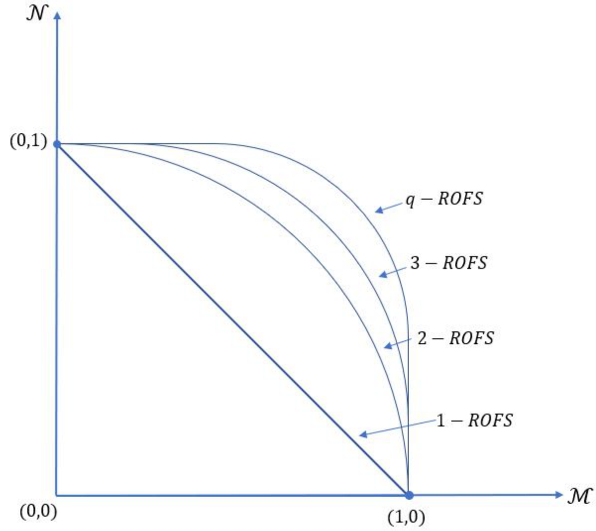

Figure 1: Representation space of q-ROFS.

Note that if $r = 0$, then $\mathcal{A}_0$ is IFS. Alternatively, CIFS can also be defined as the following expression. Let $L^* = \Big\{\langle a, b\rangle | a, b \in [0, 1] \text{ and } a + b \leq 1\Big\}$, then $\mathcal{A}_r$ could be expressed in the form $\mathcal{A}_r = \Big\{\langle x, O_r(\mathcal{M}_{\mathcal{A}}, \mathcal{N}_{\mathcal{A}})\rangle | x \in X\Big\}$, where,

$$O_r(\mathcal{M}_{\mathcal{A}}, \mathcal{N}_{\mathcal{A}}) = \Big\{\langle a, b\rangle | a, b \in [0, 1] \text{ and } \sqrt{(\mathcal{M}_{\mathcal{A}}(x) - a)^2 + (\mathcal{N}_{\mathcal{A}}(x) - b)^2} \leq r\Big\} \cap L^*.$$

3 Circular q-Rung Orthopair Fuzzy Sets

In this section, we shall define a generalization of CIFS for non-empty set X based on q-ROFS which we call Cq-ROFS. Furthermore, we introduce the relations and operations over Cq-ROFS.

Definition 3.1. *A circular q-rung orthopair fuzzy set $\mathcal{A}^*_r$ (denoted Cq-ROFS $\mathcal{A}^*_r$) in X is defined as an object of the form:*

$$\mathcal{A}^*_r = \Big\{\langle x, \mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x); r\rangle | x \in X\Big\},$$

where $\mathcal{M}_{\mathcal{A}^*} : X \to [0, 1]$ and $\mathcal{N}_{\mathcal{A}^*} : X \to [0, 1]$ denoted the degrees of membership and non-membership of the element $x \in X$, respectively, such that $0 \leq \mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) \leq 1$ for positive integer $q \geq 1$. $r \in [0, \sqrt{2}]$ is the radius of the circle around the point $(\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x))$.

The value of $\mathcal{H}_{\mathcal{A}^*}(x) = \sqrt[q]{1 - \mathcal{M}^q_{\mathcal{A}^*}(x) - \mathcal{N}^q_{\mathcal{A}^*}(x)}$ is the hesitancy degree. The collection of all Cq-ROFSs is expressed by Cq-ROFS(q, X).

Note that, if $q = 1$ then Cq-ROFS$(1, X)$=CIFS(X). The geometric interpretation of Cq-ROFS(q, X) is depicted in Figure 2.

Remark 3.1. *It is evident for every $a, b \in [0, 1]$ with $r \in [0, \sqrt{2}]$, and $q \geq 1$, if $0 \leq a + b \leq 1$ then we get $0 \leq a^q + b^q \leq 1$, since $a^q \leq a$ and $b^q \leq b$. This implies that, if $\mathcal{A}^*_r \in CIFS(X)$ then also $\mathcal{A}^*_r \in Cq - ROFS(q, X)$.*

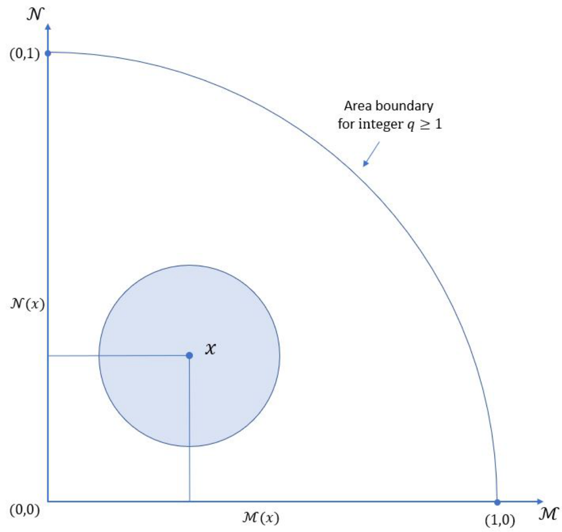

Figure 2: Geometrical interpretation of Cq-ROFS(q, X) for integer $q \geq 1$.

The comparisons between Cq-ROFS and other different sets (i.e., q-ROFS, CIFS and IFS) are presented in Table 1. Notably, Cq-ROFS demonstrates generality by encompassing not only the case of uncertainty and indeterminacy, as in IFS $(a + b \leq 1)$, but also includes q-ROFS $(a + b > 1)$ and CIFS, considering an imprecise region around the point $(\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x))$ with radius $r \geq 0$.

Table 1: The comparisons of Cq-ROFS with different sets.

	Cq-ROFS	q-ROFS	CIFS	IFS
$a + b \leq 1$	Yes	Yes	Yes	Yes
$a + b > 1$	Yes	Yes	No	No
$r \geq 0$	Yes	No	Yes	No

In the following, the definition of relations over Cq-ROFS is presented, followed by the definition of operations over Cq-ROFS.

Definition 3.2. *Let $\mathcal{A}_r^*, \mathcal{B}_s^* \in Cq-ROFS$ for $r, s \in [0, \sqrt{2}]$. The relations between $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ are defined as follows:*

1. $\mathcal{A}_r^* \subset_\nu \mathcal{B}_s^* \Leftrightarrow (\forall x \in X)(r = s)$ *and one of the following condition arises,*

$$\mathcal{M}_{\mathcal{A}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x) \textit{ and } \mathcal{N}_{\mathcal{A}^*}(x) \geq \mathcal{N}_{\mathcal{B}^*}(x),$$
$$\mathcal{M}_{\mathcal{A}^*}(x) \leq \mathcal{M}_{\mathcal{B}^*}(x) \textit{ and } \mathcal{N}_{\mathcal{A}^*}(x) > \mathcal{N}_{\mathcal{B}^*}(x),$$
$$\mathcal{M}_{\mathcal{A}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x) \textit{ and } \mathcal{N}_{\mathcal{A}^*}(x) > \mathcal{N}_{\mathcal{B}^*}(x).$$

2. $\mathcal{A}_r^* \subset_\rho \mathcal{B}_s^* \Leftrightarrow (\forall x \in X)(r < s)$ *and* $\mathcal{M}_{\mathcal{A}^*}(x) = \mathcal{M}_{\mathcal{B}^*}(x)$ *and* $\mathcal{N}_{\mathcal{A}^*}(x) = \mathcal{N}_{\mathcal{B}^*}(x)$.

3. $\mathcal{A}_r^* \subset \mathcal{B}_s^* \Leftrightarrow (\forall x \in X)(r < s)$ *and one of the following condition arises,*

$$\mathcal{M}_{\mathcal{A}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x) \textit{ and } \mathcal{N}_{\mathcal{A}^*}(x) \geq \mathcal{N}_{\mathcal{B}^*}(x),$$
$$\mathcal{M}_{\mathcal{A}^*}(x) \leq \mathcal{M}_{\mathcal{B}^*}(x) \textit{ and } \mathcal{N}_{\mathcal{A}^*}(x) > \mathcal{N}_{\mathcal{B}^*}(x),$$
$$\mathcal{M}_{\mathcal{A}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x) \textit{ and } \mathcal{N}_{\mathcal{A}^*}(x) > \mathcal{N}_{\mathcal{B}^*}(x).$$

4. $\mathcal{A}_r^* =_\nu \mathcal{B}_s^* \Leftrightarrow (\forall x \in X)$ *and* $\mathcal{M}_{\mathcal{A}^*}(x) = \mathcal{M}_{\mathcal{B}^*}(x)$ *and* $\mathcal{N}_{\mathcal{A}^*}(x) = \mathcal{N}_{\mathcal{B}^*}(x)$.
5. $\mathcal{A}_r^* =_\rho \mathcal{B}_s^* \Leftrightarrow r = s$.
6. $\mathcal{A}_r^* = \mathcal{A}_s^* \Leftrightarrow (\forall x \in X)(r = s)$ *and* $\mathcal{M}_{\mathcal{A}^*}(x) = \mathcal{M}_{\mathcal{B}^*}(x)$ *and* $\mathcal{N}_{\mathcal{A}^*}(x) = \mathcal{N}_{\mathcal{B}^*}(x)$.

Before defining the general operations for Cq-ROFS, we first establish the operations pertaining to the radius of Cq-ROFS. In a previous work by Atanassov [9], the radius operations for CIFS were defined based on max and min, where the radius $r, s \in [0, 1]$. In this work, we propose not only the max and min operations but also the algebraic product and algebraic sum as additional operations for the radius in the Cq-ROFS. Furthermore, we extend the range of the radius $r, s \in [0, \sqrt{2}]$.

Definition 3.3. *Let $r, s \in [0, \sqrt{2}]$ and q be a positive integer. The algebraic product $\otimes$ and algebraic sum $\oplus$ on r and s for Cq-ROFS are defined as follows:*

$$\otimes(r, s) = \frac{rs}{\sqrt{2}} \text{ and } \oplus(r, s) = \left(r^q + s^q - \left(\frac{rs}{\sqrt{2}}\right)^q\right)^{\frac{1}{q}}.$$

It can be observed that the above radius operations satisfy the boundary condition. This is further supported by the following theorem, which provides proof for these operations.

Theorem 3.1. *Let $r, s \in [0, \sqrt{2}]$, then $\otimes(r, s), \oplus(r, s) \in [0, \sqrt{2}]$.*

Proof. To prove these operations, we must show that for $r, s \in [0, \sqrt{2}]$ and positive integer $q \geq 1$, the boundary condition for $\otimes(r, s), \oplus(r, s) \in [0, \sqrt{2}]$ is valid. It is clear that for $r, s = 0$, then Cq-ROFSs are reduced to q-ROFSs, which satisfy the boundary condition. For $r, s = \sqrt{2}$, then $\otimes(r, s) = \frac{rs}{\sqrt{2}} = \sqrt{2}$. In the case of $\oplus(r, s)$, we must prove that $r^q + s^q - \left(\frac{rs}{\sqrt{2}}\right)^q \leq \sqrt{2}^q$. Using the contradiction, suppose it is true for $r^q + s^q - \left(\frac{rs}{\sqrt{2}}\right)^q > \sqrt{2}^q$ such that:

$$\begin{aligned} r^q + s^q - \left(\frac{rs}{\sqrt{2}}\right)^q - \sqrt{2}^q &> 0, \\ \sqrt{2}^q r^q + \sqrt{2}^q s^q - r^q s^q - \sqrt{2}^{2q} &> 0, \\ \left(r^q - \sqrt{2}^q\right)\left(\sqrt{2}^q - s^q\right) &> 0. \end{aligned}$$

For positive integer $q \geq 1$ and $0 \leq r, s \leq \sqrt{2}$ then, we obtain $\left(r^q - \sqrt{2}^q\right)\left(\sqrt{2}^q - s^q\right) \leq 0$. Therefore, it is contradicted, hence, $r^q + s^q - \left(\frac{rs}{\sqrt{2}}\right)^q \leq \sqrt{2}^q$. This implies,

$$0 \leq \oplus(r, s) = \left(r^q + s^q - \left(\frac{rs}{\sqrt{2}}\right)^q\right)^{\frac{1}{q}} \leq \left(\sqrt{2}^q\right)^{\frac{1}{q}} = \sqrt{2}.$$

□

Remark 3.2. *The max and min operations are rather straightforward. For $r, s \in [0, \sqrt{2}]$, we have max(r, s) and min(r, s) which satisfy the boundary condition (see [9]).*

Next, we define the general operations over Cq-ROFSs based on the radius operations in Definition 3.3.

Definition 3.4. *Let $\mathcal{A}_r^*, \mathcal{B}_s^* \in Cq\text{-}ROFS$ with $r, s \in [0, \sqrt{2}]$ and positive integer $q \geq 1$. For any $x \in X$ where $\propto \in \{min, max, \oplus, \otimes\}$ represents the radius operator, the operations between $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ can be defined as follows:*

1. $\neg\mathcal{A}_r^* = \Big\{\langle x, \mathcal{N}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{A}^*}(x); r\rangle\Big\}$.
2. $\mathcal{A}_r^* \cap_\propto \mathcal{B}_s^* = \Big\{\langle x, \min\Big\{\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{B}^*}(x)\Big\}, \max\Big\{\mathcal{N}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{B}^*}(x)\Big\}; \propto(r, s)\rangle\Big\}$.
3. $\mathcal{A}_r^* \cup_\propto \mathcal{B}_s^* = \Big\{\langle x, \max\Big\{\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{B}^*}(x)\Big\}, \min\Big\{\mathcal{N}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{B}^*}(x)\Big\}; \propto(r, s)\rangle\Big\}$.
4. $\mathcal{A}_r^* +_\propto \mathcal{B}_s^* = \Big\{\langle x, \Big(\mathcal{M}_{\mathcal{A}^*}^q(x) + \mathcal{M}_{\mathcal{B}^*}^q(x) - \mathcal{M}_{\mathcal{A}^*}^q(x)\mathcal{M}_{\mathcal{B}^*}^q(x)\Big)^{\frac{1}{q}}, \mathcal{N}_{\mathcal{A}^*}(x) \cdot \mathcal{N}_{\mathcal{B}^*}(x); \propto(r, s)\rangle\Big\}$.
5. $\mathcal{A}_r^* \circ_\propto \mathcal{B}_s^* = \Big\{\langle x, \mathcal{M}_{\mathcal{A}^*}(x) \cdot \mathcal{M}_{\mathcal{B}^*}(x), \Big(\mathcal{N}_{\mathcal{A}^*}^q(x) + \mathcal{N}_{\mathcal{B}^*}^q(x) - \mathcal{N}_{\mathcal{A}^*}^q(x)\mathcal{N}_{\mathcal{B}^*}^q(x)\Big)^{\frac{1}{q}}; \propto(r, s)\rangle\Big\}$.

Theorem 3.2. *Let $\mathcal{A}_r^*, \mathcal{B}_s^* \in Cq\text{-}ROFS$ with $r, s \in [0, \sqrt{2}]$. For all $\phi \in \{\cap, \cup, +, \circ\}$ and $\propto \in \{\min, \max, \otimes, \oplus\}$, then $\neg\mathcal{A}_r^*$, $\mathcal{A}_r^*\phi_\propto\mathcal{B}_s^* \in Cq\text{-}ROFS$.*

Proof. Here, we focus only on the operations related to $\mathcal{M}(x)$ and $\mathcal{N}(x)$ for every $x \in X$. The proof for the radius operations is already covered in Theorem 3.1.

1. For $\neg\mathcal{A}_r^*$, the proof is straightforward.
2. For $\mathcal{A}_r^* \cap_\propto \mathcal{B}_s^*$, suppose
$$\max\Big\{\mathcal{N}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{B}^*}(x)\Big\} = \mathcal{N}_{\mathcal{A}^*}(x) \quad \text{and} \quad \min\Big\{\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{B}^*}(x)\Big\} \leq \mathcal{M}_{\mathcal{A}^*}(x),$$
then we have:
$$\begin{aligned} 0 \leq \Big(\mathcal{M}_{\mathcal{A}_r^* \cap_\propto \mathcal{B}_s^*}(x)\Big)^q + \Big(\mathcal{N}_{\mathcal{A}_r^* \cap_\propto \mathcal{B}_s^*}(x)\Big)^q &= \Big(\min\Big\{\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{B}^*}(x)\Big\}\Big)^q + \Big(\mathcal{N}_{\mathcal{A}^*}(x)\Big)^q \\ &\leq (\mathcal{M}_{\mathcal{A}^*}(x))^q + (\mathcal{N}_{\mathcal{A}^*}(x))^q \leq 1, \end{aligned}$$
which satisfy the boundary condition. Similarly, if $\max\Big\{\mathcal{N}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{B}^*}(x)\Big\} = \mathcal{N}_{\mathcal{B}^*}(x)$ and $\min\Big\{\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{B}^*}(x)\Big\} \leq \mathcal{M}_{\mathcal{B}^*}(x)$, then we obtain $0 \leq (\mathcal{M}_{\mathcal{B}^*}(x))^q + (\mathcal{N}_{\mathcal{B}^*}(x))^q \leq 1$.
3. We can prove in the same way for $\mathcal{A}_r^* \cup_\propto \mathcal{B}_s^*$.
4. Next, for operations $+_\propto$ and $\circ_\propto$, we have:
$$\begin{aligned} 0 &\leq \Big(\mathcal{M}_{\mathcal{A}_r^* +_\propto \mathcal{B}_s^*}(x)\Big)^q + \Big(\mathcal{N}_{\mathcal{A}_r^* +_\propto \mathcal{B}_s^*}(x)\Big)^q, \\ &= \mathcal{M}_{\mathcal{A}^*}^q(x) + \mathcal{M}_{\mathcal{B}^*}^q(x) - \mathcal{M}_{\mathcal{A}^*}^q(x).\mathcal{M}_{\mathcal{B}^*}^q(x) + \mathcal{N}_{\mathcal{A}^*}^q(x).\mathcal{N}_{\mathcal{B}^*}^q(x), \\ &\leq \mathcal{M}_{\mathcal{A}^*}^q(x) + \mathcal{M}_{\mathcal{B}^*}^q(x) - \mathcal{M}_{\mathcal{A}^*}^q(x).\mathcal{M}_{\mathcal{B}^*}^q(x) + \Big(1 - \mathcal{M}_{\mathcal{A}^*}^q(x)\Big).\Big(1 - \mathcal{M}_{\mathcal{B}^*}^q(x)\Big) = 1. \end{aligned}$$
5. Likewise, $\mathcal{A}_r^* \circ_\propto \mathcal{B}_s^*$ can be proved analogously.

This completes the proof. □

Note that the operations defined above assume that the same q is considered (for any $q \geq 1$). However, in the case of different rungs, we can implement $q = \max(q_1, q_2)$ where q_1 and q_2 are elements of the set of rungs. This was suggested by Yager [29] for the case of q-ROFS to ensure inclusivity of all rungs under consideration. Similarly, this also applies to the case of Cq-ROFS.

Remark 3.3. *Assume that $\mathcal{A}_r^* \in Cq\text{-}ROFS(q_1, X)$ and $\mathcal{B}_s^* \in Cq\text{-}ROFS(q_2, X)$ have different rungs q_1 and q_2. Any operations performed on $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ result in Cq-ROFS(q, X), where $q = \max(q_1, q_2)$.*

The next discussion concerns the algebraic properties that apply to these operations. The properties are evidenced in, among others, idempotency, inclusion, and absorption.

Theorem 3.3. (*Idempotency*) *For any Cq-ROFSs $\mathcal{A}_r^*$ with $r \in [0, \sqrt{2}]$, given that $\phi \in \{\cap, \cup, +, \circ\}$ and $\propto \in \{\min, \max, \otimes, \oplus\}$, then $\mathcal{A}_r^* \phi_\propto \mathcal{A}_r^* = \mathcal{A}_r^*$.*

Proof. The proof follows immediately from Definitions 3.3 and 3.4. □

Lemma 3.1. *Let $r, s \in [0, \sqrt{2}]$ and positive integer $q \geq 1$, then the following inequalities hold:*

1. *$\otimes(r, s) < r$ or s.*
2. *$\oplus(r, s) > r$ or s.*

Proof. We prove Lemma 3.1 by contradiction.

1. Suppose $\otimes(r, s) = \dfrac{rs}{\sqrt{2}} > r$, then:
$$\frac{rs}{\sqrt{2}} - r > 0,$$
$$\frac{r}{\sqrt{2}}\left(s - \sqrt{2}\right) > 0.$$
Note that, since $s \in [0, \sqrt{2}]$, then we have $(s - \sqrt{2}) \leq 0$. Therefore, the assumption is wrong and $\otimes(r, s) < r$. Similarly, $\otimes(r, s) < s$ can be proved in the same way.

2. Suppose $\oplus(r, s) < r = \left(r^q + s^q - \left(\dfrac{rs}{\sqrt{2}}\right)^q\right)^{\frac{1}{q}} < r$, then:
$$r^q + s^q - \left(\frac{rs}{\sqrt{2}}\right)^q < r^q,$$
$$\frac{s^q}{\sqrt{2}^q}\left(\sqrt{2}^q - r^q\right) < 0.$$
Since $r \in [0, \sqrt{2}]$, then $\left(\sqrt{2}^q - r^q\right) \geq 0$. Hence, the assumption is wrong and $\oplus(r, s) > r$. Similarly, $\oplus(r, s) > s$ can be proved analogously.

□

Lemma 3.1 is used to determine the consistency of inclusion properties in Cq-ROFS.

Theorem 3.4. (*Inclusion*) *For every two Cq-ROFSs $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ with $r, s \in [0, \sqrt{2}]$ and $\propto \in \{\min, \max, \otimes, \oplus\}$, we have:*

1. *if $\mathcal{A}_r^* \subset \mathcal{B}_s^*$, then $\mathcal{A}_r^* \circ_\propto \mathcal{B}_s^* \subset \mathcal{B}_s^*$,*
2. *if $\mathcal{A}_r^* \subset \mathcal{B}_s^*$, then $\mathcal{A}_r^* +_\propto \mathcal{B}_s^* \subset \mathcal{B}_s^*$.*

Proof. Let $\mathcal{A}_r^*, \mathcal{B}_s^* \in Cq\text{-}ROFS$ with $r, s \in [0, \sqrt{2}]$ and $\mathcal{A}_r^* \subset \mathcal{B}_s^*$ such that $(\forall x \in X)(r < s)$ and satisfy $\mathcal{M}_{\mathcal{A}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x)$ and $\mathcal{N}_{\mathcal{A}^*}(x) > \mathcal{N}_{\mathcal{B}^*}(x)$. For operation $\mathcal{A}_r^* \circ_\propto \mathcal{B}_s^*$, the following membership values are obtained, $\mathcal{M}_{\mathcal{A}^*}(x).\mathcal{M}_{\mathcal{B}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x)$. Analogously, this also applies to non-membership values, where we obtained $\left[\mathcal{N}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{B}^*}(x) - \mathcal{N}^q_{\mathcal{A}^*}(x).\mathcal{N}^q_{\mathcal{B}^*}(x)\right]^{\frac{1}{q}} > \mathcal{N}_{\mathcal{B}^*}(x)$. For the radius operations, the obtained value for each $\propto \in \{\min, \max, \otimes, \oplus\}$ can be proved by Definition 3.2. Therefore, it is clear that $\mathcal{A}_r^* +_\propto \mathcal{B}_s^* \subset \mathcal{B}_s^*$ and similarly $\mathcal{A}_r^* \circ_\propto \mathcal{B}_s^* \subset \mathcal{B}_s^*$. □

Lemma 3.2. *Let $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ are Cq-ROFS with $r, s \in [0, \sqrt{2}]$ and positive integer $q \geq 1$, then the following relations hold:*

1. $\mathcal{A}_r^* \subset (\mathcal{A}_r^* \cup_{\max} \mathcal{B}_s^*) \subset_\rho (\mathcal{A}_r^* \cup_\oplus \mathcal{B}_s^*)\,; \mathcal{A}_r^* \subset (\mathcal{A}_r^* +_{\max} \mathcal{B}_s^*) \subset_\rho (\mathcal{A}_r^* +_\oplus \mathcal{B}_s^*)$,
2. $(\mathcal{A}_r^* \cap_\otimes \mathcal{B}_s^*) \subset_\rho (\mathcal{A}_r^* \cap_{\min} \mathcal{B}_s^*) \subset \mathcal{A}_r^*; (\mathcal{A}_r^* \circ_\otimes \mathcal{B}_s^*) \subset_\rho (\mathcal{A}_r^* \circ_{\min} \mathcal{B}_s^*) \subset \mathcal{A}_r^*$.

Proof. The validity of Lemma 3.2 follows from the Definition 3.4 and Lemma 3.1.
Since for $r, s \in [0, \sqrt{2}]$ and positive integer $q \geq 1$ then we have:

$$r \leq \max(r, s) \leq r^q + s^q - \left(\frac{rs}{\sqrt{2}}\right)^q.$$

Analogously:

$$\frac{rs}{\sqrt{2}} \leq \min(r, s) \leq r.$$

The proof is now completed. □

Theorem 3.5. (*Absorption*) *For every two Cq-ROFS, $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ with $r, s \in [0, \sqrt{2}]$, $\phi \in \{\cup, +\}$, $\varphi \in \{\cap, \circ\}$ and $\propto \in \{\min, \max, \otimes, \oplus\}$, then:*

1. $\mathcal{A}_r^* \circ_\propto (\mathcal{A}_r^* \phi_{\max} \mathcal{B}_s^*) \subset \mathcal{A}_r^* \phi_{\max} \mathcal{B}_s^*; \mathcal{A}_r^* \circ_\propto (\mathcal{A}_r^* \phi_\oplus \mathcal{B}_s^*) \subset \mathcal{A}_r^* \phi_\oplus \mathcal{B}_s^*$,
2. $\mathcal{A}_r^* +_\propto (\mathcal{A}_r^* \phi_{\max} \mathcal{B}_s^*) \subset \mathcal{A}_r^* \phi_{\max} \mathcal{B}_s^*; \mathcal{A}_r^* +_\propto (\mathcal{A}_r^* \phi_\oplus \mathcal{B}_s^*) \subset \mathcal{A}_r^* \phi_\oplus \mathcal{B}_s^*$,
3. $(\mathcal{A}_r^* \varphi_\otimes \mathcal{B}_s^*) \circ_\propto \mathcal{A}_r^* \subset \mathcal{A}_r^*; (\mathcal{A}_r^* \varphi_{\min} \mathcal{B}_s^*) \circ_\propto \mathcal{A}_r^* \subset \mathcal{A}_r^*$,
4. $(\mathcal{A}_r^* \varphi_\otimes \mathcal{B}_s^*) +_\propto \mathcal{A}_r^* \subset \mathcal{A}_r^*; (\mathcal{A}_r^* \varphi_{\min} \mathcal{B}_s^*) +_\propto \mathcal{A}_r^* \subset \mathcal{A}_r^*$.

Proof. The proof can be demonstrated by utilizing Lemma 3.2 and Theorem 3.4. □

4 Some Modal Operators Over Cq-ROFS

In this section, some of modal operators are defined for Cq-ROFS over the universal set X. Previously, Atanassov [9] defined "necessity" and "possibility" followed by the definition of modal operators for CIFS. Other studies have also defined the type of modal operators that applies to q-ROFS. In the following, based on the existing definitions, we define the modal operators for Cq-ROFS accompanied by their corresponding properties.

Definition 4.1. *For any Cq-ROFS $\mathcal{A}_r^*$ with positive integer $q \geq 1$ and $\lambda, \gamma \in [0,1]$ for $\lambda + \gamma \leq 1$ be any real number, then some of modal operators over Cq-ROFS are defined as follows:*

1. $\Box\mathcal{A}_r^* = \left\{\langle x, \mathcal{M}_{\mathcal{A}^*}(x), (1-\mathcal{M}_{\mathcal{A}^*}^q(x))^{\frac{1}{q}}; r\rangle | x \in X\right\}$.
2. $\Diamond\mathcal{A}_r^* = \left\{\langle x, (1-\mathcal{N}_{\mathcal{A}^*}^q(x))^{\frac{1}{q}}, \mathcal{N}_{\mathcal{A}^*}(x); r\rangle | x \in X\right\}$.
3. $D_\lambda(\mathcal{A}_r^*) = \left\{\langle x, (\mathcal{M}_{\mathcal{A}^*}^q(x) + \lambda.\mathcal{H}_{\mathcal{A}^*}^q(x))^{\frac{1}{q}}, (\mathcal{N}_{\mathcal{A}^*}^q(x) + (1-\lambda).\mathcal{H}_{\mathcal{A}^*}^q(x))^{\frac{1}{q}}; r\rangle | x \in X\right\}$.
4. $F_{\lambda,\gamma}(\mathcal{A}_r^*) = \left\{\langle x, (\mathcal{M}_{\mathcal{A}^*}^q(x) + \lambda.\mathcal{H}_{\mathcal{A}^*}^q(x))^{\frac{1}{q}}, (\mathcal{N}_{\mathcal{A}^*}^q(x) + \gamma.\mathcal{H}_{\mathcal{A}^*}^q(x))^{\frac{1}{q}}; r\rangle | x \in X\right\}$.
5. $G_{\lambda,\gamma}(\mathcal{A}_r^*) = \left\{\langle x, \lambda^{\frac{1}{q}}.\mathcal{M}_{\mathcal{A}^*}(x), \gamma^{\frac{1}{q}}.\mathcal{N}_{\mathcal{A}^*}(x); r\rangle | s \in X\right\}$.
6. $H_{\lambda,\gamma}(\mathcal{A}_r^*) = \left\{\langle x, \lambda^{\frac{1}{q}}.\mathcal{M}_{\mathcal{A}^*}(x), (\mathcal{N}_{\mathcal{A}^*}^q(x) + \gamma.\mathcal{H}_{\mathcal{A}^*}^q(x))^{\frac{1}{q}}; r\rangle | s \in X\right\}$.
7. $J_{\lambda,\gamma}(\mathcal{A}_r^*) = \left\{\langle x, (\mathcal{M}_{\mathcal{A}^*}^q(x) + \lambda.\mathcal{H}_{\mathcal{A}^*}^q(x))^{\frac{1}{q}}, \gamma^{\frac{1}{q}}.\mathcal{N}_{\mathcal{A}^*}(x); r\rangle | s \in X\right\}$.

Additionally, it must be confirmed that the modal operator specified in Definition 4.1 is also Cq-ROFS.

Theorem 4.1. *The Cq-ROFS operations defined by Definition 4.1 are also Cq-ROFS.*

Proof. For $\mathcal{A}_r^* \in Cq\text{-}ROFS$ such that $\mathcal{A}_r^* = \left\{\langle x, \mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x); r\rangle | x \in X\right\}$, integer $q \geq 1$ and $\lambda, \gamma \in [0,1]$ for $\lambda + \gamma \leq 1$, then for each $x \in X$:

1. Since $0 \leq \mathcal{M}_{\mathcal{A}^*}(x) \leq 1$ and $q \geq 1$, then:

$$0 \leq 1 - \left(1 - \mathcal{M}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}} \leq 1.$$

   Therefore,

$$\begin{aligned}\mathcal{M}_{\Box\mathcal{A}^*}^q(x) + \mathcal{N}_{\Box\mathcal{A}^*}^q(x) &= [\mathcal{M}_{\mathcal{A}^*}(x)]^q + \left[\left(1 - \mathcal{M}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}\right]^q \\ &= \mathcal{M}_{\mathcal{A}^*}^q(x) + \left(1 - \mathcal{M}_{\mathcal{A}^*}^q(x)\right) = 1.\end{aligned}$$

2. The operator $\Diamond\mathcal{A}_r^*$ can be proved analogously.

3. For any real number $\lambda \in [0,1]$ and Cq-ROFS $\mathcal{A}_r$, we have $0 \leq \mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x) \leq 1$ and $0 \leq \mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) \leq 1$ which implies that $\mathcal{H}^q_{\mathcal{A}^*}(x) \leq 1$. Hence,

$$\mathcal{M}_{D_\lambda(\mathcal{A}^*_r)}(x) = \left(\mathcal{M}^q_{\mathcal{A}^*}(x) + \lambda.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}} \text{ and } \mathcal{N}_{D_\lambda(\mathcal{A}^*_r)}(x) = \left(\mathcal{N}^q_{\mathcal{A}^*}(x) + (1-\lambda).\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}}.$$

Furthermore:

$$\begin{aligned}\mathcal{M}^q_{D_\lambda(\mathcal{A}^*_r)}(x) + \mathcal{N}^q_{D_\lambda(\mathcal{A}^*_r)}(x) &= \left[\left(\mathcal{M}^q_{\mathcal{A}^*}(x) + \lambda.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}}\right]^q + \left[\left(\mathcal{N}^q_{\mathcal{A}^*}(x) + (1-\lambda).\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}}\right]^q \\ &= \mathcal{M}^q_{\mathcal{A}^*}(x) + \lambda.\mathcal{H}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) + (1-\lambda).\mathcal{H}^q_{\mathcal{A}^*}(x) \\ &= \mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) + \mathcal{H}^q_{\mathcal{A}^*}(x) = 1.\end{aligned}$$

4. For any real number $\lambda, \gamma \in [0,1]$ where $\lambda + \gamma \leq 1$ and Cq-ROFS $\mathcal{A}^*_r$, we have $0 \leq \mathcal{M}_{\mathcal{A}^*}(x) \leq 1$ and $0 \leq \mathcal{N}_{\mathcal{A}^*}(x) \leq 1$ and $0 \leq \mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) \leq 1$ which implies that $\mathcal{H}^q_{\mathcal{A}^*}(x) \leq 1$. Hence,

$$\mathcal{M}_{F_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) = \left(\mathcal{M}^q_{\mathcal{A}^*}(x) + \lambda.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}} \text{ and } \mathcal{N}_{F_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) = \left(\mathcal{N}^q_{\mathcal{A}^*}(x) + \gamma.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}}.$$

Furthermore:

$$\begin{aligned}\mathcal{M}^q_{F_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) + \mathcal{N}^q_{F_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) &= \left[\left(\mathcal{M}^q_{\mathcal{A}^*}(x) + \lambda.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}}\right]^q + \left[\left(\mathcal{N}^q_{\mathcal{A}^*}(x) + \gamma.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}}\right]^q \\ &= \mathcal{M}^q_{\mathcal{A}^*}(x) + \lambda.\mathcal{H}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) + \gamma.\mathcal{H}^q_{\mathcal{A}^*}(x) \\ &= \mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) + (\lambda+\gamma)\mathcal{H}^q_{\mathcal{A}^*}(x) \leq 1.\end{aligned}$$

5. Analogously to Proof 4), where $\mathcal{M}_{G_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) = \lambda^{\frac{1}{q}}.\mathcal{M}_{\mathcal{A}^*}(x)$ and $\mathcal{N}_{G_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) = \gamma^{\frac{1}{q}}.\mathcal{N}_{\mathcal{A}^*}(x)$, then we obtain:

$$\begin{aligned}\mathcal{M}_{G_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) + \mathcal{N}_{G_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) &= \left[\lambda^{\frac{1}{q}}.\mathcal{M}_{\mathcal{A}^*}(x)\right]^q + \left[\gamma^{\frac{1}{q}}.\mathcal{N}_{\mathcal{A}^*}(x)\right]^q \\ &= \lambda.\mathcal{M}^q_{\mathcal{A}^*}(x) + \gamma.\mathcal{N}^q_{\mathcal{A}^*}(x) \\ &\leq \mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) = 1.\end{aligned}$$

6. Likewise,

$$\mathcal{M}_{H_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) = \lambda^{\frac{1}{q}}.\mathcal{M}_{\mathcal{A}^*}(x) \text{ and } \mathcal{N}_{H_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) = \left(\mathcal{N}^q_{\mathcal{A}^*}(x) + \gamma.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}},$$

then we obtained:

$$\begin{aligned}\mathcal{M}_{H_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) + \mathcal{N}_{H_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) &= \left[\lambda^{\frac{1}{q}}.\mathcal{M}_{\mathcal{A}^*}(x)\right]^q + \left[\left(\mathcal{N}^q_{\mathcal{A}^*}(x) + \gamma.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}}\right]^q \\ &= \lambda.\mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) + \gamma.\mathcal{H}^q_{\mathcal{A}^*}(x) \\ &\leq \mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) + \mathcal{H}^q_{\mathcal{A}^*}(x) = 1.\end{aligned}$$

7. Let,

$$\mathcal{M}_{J_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) = \left(\mathcal{M}^q_{\mathcal{A}^*}(x) + \lambda.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}} \text{ and } \mathcal{N}_{J_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) = \gamma^{\frac{1}{q}}.\mathcal{N}_{\mathcal{A}^*}(x).$$

Then we obtain:

$$\begin{aligned}\mathcal{M}_{J_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) + \mathcal{N}_{J_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) &= \left[\left(\mathcal{M}^q_{\mathcal{A}^*}(x) + \lambda.\mathcal{H}^q_{\mathcal{A}^*}(x)\right)^{\frac{1}{q}}\right]^q + \left[\gamma^{\frac{1}{q}}.\mathcal{N}_{\mathcal{A}^*}(x)\right]^q \\ &= \mathcal{M}^q_{\mathcal{A}^*}(x) + \lambda.\mathcal{H}^q_{\mathcal{A}^*}(x) + \gamma.\mathcal{N}^q_{\mathcal{A}^*}(x) \\ &\leq \mathcal{M}^q_{\mathcal{A}^*}(x) + \mathcal{H}^q_{\mathcal{A}^*}(x) + \mathcal{N}^q_{\mathcal{A}^*}(x) = 1.\end{aligned}$$

Hence, it is proven that the operations defined in Definiton 4.1 are also Cq-ROFS. $\square$

In the following, the relations, and basic properties of the modal operators for Cq-ROFS are presented.

Theorem 4.2. *For any Cq-ROFS $\mathcal{A}_r^*$ and every $\lambda, \gamma \in [0,1]$:*

1) $\lambda \leq \gamma \Rightarrow D_\lambda(\mathcal{A}_r^*) \subseteq D_\gamma(\mathcal{A}_r^*)$.
2) $D_0(\mathcal{A}_r^*) = \Box\mathcal{A}_r^*$ *and* $D_1(\mathcal{A}_r^*) = \Diamond\mathcal{A}_r^*$.
3) $0 \leq \eta \leq \lambda \Rightarrow F_{\eta,\gamma}(\mathcal{A}_r^*) \subseteq F_{\lambda,\gamma}(\mathcal{A}_r^*)$.
4) $0 \leq \eta \leq \gamma \Rightarrow F_{\lambda,\eta}(\mathcal{A}_r^*) \supseteq F_{\lambda,\gamma}(\mathcal{A}_r^*)$.
5) $F_{\lambda,1-\lambda}(\mathcal{A}_r^*) = D_\lambda(\mathcal{A}_r^*)$,
6) $F_{0,1}(\mathcal{A}_r^*) = \Box\mathcal{A}_r^*$ *and* $F_{1,0}(\mathcal{A}_r^*) = \Diamond\mathcal{A}_r^*$.
7) $\lambda \leq \eta \Rightarrow G_{\lambda,\gamma}(\mathcal{A}_r^*) \subseteq G_{\eta,\gamma}(\mathcal{A}_r^*)$.
8) $\gamma \leq \eta \Rightarrow G_{\lambda,\gamma}(\mathcal{A}_r^*) \supseteq G_{\lambda,\eta}(\mathcal{A}_r^*)$.

Proof. Here, proofs are given only in a few parts, with the assumption that the other parts are analogous for every $x \in X$.

1) For the Definition 4.1 and $\lambda, \gamma \in [0,1]$ we get:

$$D_\lambda(\mathcal{A}_r^*) = \left\{x, \left(\mathcal{M}_{\mathcal{A}^*}^q(x) + \lambda.\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}, \left(\mathcal{N}_{\mathcal{A}^*}^q(x) + (1-\lambda).\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}; r\right\}, \text{ and}$$
$$D_\gamma(\mathcal{A}_r^*) = \left\{x, \left(\mathcal{M}_{\mathcal{A}^*}^q(x) + \gamma.\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}, \left(\mathcal{N}_{\mathcal{A}^*}^q(x) + (1-\gamma).\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}; r\right\}.$$

Since $\lambda \leq \gamma$ then $\lambda.\mathcal{H}_{\mathcal{A}^*}^q(x) \leq \gamma.\mathcal{H}_{\mathcal{A}^*}^q(x)$ and therefore, it is proven.

2)

$$\begin{aligned} D_0(\mathcal{A}_r^*) &= \left\{x, \left(\mathcal{M}_{\mathcal{A}^*}^q(x) + 0.\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}, \left(\mathcal{N}_{\mathcal{A}^*}^q(x) + (1-0).\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}; r\right\} \\ &= \left\{x, \mathcal{M}_{\mathcal{A}^*}(x), \left(\mathcal{N}_{\mathcal{A}^*}^q(x) + 1 - \mathcal{M}_{\mathcal{A}^*}^q(x) - \mathcal{N}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}; r\right\} \\ &= \left\{x, \mathcal{M}_{\mathcal{A}^*}(x), \left(1 - \mathcal{M}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}; r\right\} = \Box\mathcal{A}_r^*. \end{aligned}$$

The same applies to $D_1(\mathcal{A}_r^*) = \Diamond\mathcal{A}_r^*$.

Proofs 3) and 4) are analogous to proof 1).

5) $F_{\lambda,1-\lambda}(\mathcal{A}_r^*) = \left\{x, \left(\mathcal{M}_{\mathcal{A}^*}^q(x) + \lambda.\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}, \left(\mathcal{N}_{\mathcal{A}^*}^q(x) + (1-\lambda).\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}; r\right\} = D_\lambda(\mathcal{A}_r^*)$.

6)

$$F_{0,1}(\mathcal{A}_r^*) = \left\{x, \left(\mathcal{M}_{\mathcal{A}^*}^q(x) + 0.\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}, \left(\mathcal{N}_{\mathcal{A}^*}^q(x) + 1.\mathcal{H}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}; r\right\}$$
$$= \left\{x, \mathcal{M}_{\mathcal{A}^*}^q(x), \left(\mathcal{N}_{\mathcal{A}^*}^q(x) + 1 - \mathcal{M}_{\mathcal{A}^*}^q(x) - \mathcal{N}_{\mathcal{A}^*}^q(x)\right)^{\frac{1}{q}}; r\right\} = \Box\mathcal{A}_r^*.$$

Analogously, $F_{1,0}(\mathcal{A}_r^*) = \Diamond\mathcal{A}_r^*$.

Proofs 7) and 8) are similar to Proof 1). Therefore, the details are omitted. □

5 Conclusion and Future Studies

This article introduces a Cq-ROFS as a new and more flexible extension of CIFS and q-ROFS. We present the operations and relations for Cq-ROFS, examine modal operators, and investigate their properties. The key features of this set are that it not only expands the space of the IFIT to provide greater flexibility in handling data but also takes into account the imprecise $\mathcal{M}$ and $\mathcal{N}$ degrees. In future research, we plan to propose additional operators for Cq-ROFS, particularly the aggregation operators and study their properties. These operators include the ordered weighted averaging (OWA) operator and its families, as well as the Sugeno integral and the Choquet integral [27, 29]. Moreover, we aim to broaden the range of applications for this set, encompassing various fields such as decision-making problems [22, 4] and geometric modeling [31]. Additionally, the expansion of this set under proximity measures, such as distance and similarity measures [5], as well as ranking methods, including score and accuracy functions [6] within the Cq-ROFS environment.

Acknowledgement The authors are thankful to the anonymous reviewers for their valuable remarks that improved the quality of the paper. Support from Ministry of Higher Education Malaysia and Universiti Malaysia Terengganu are gratefully acknowledged.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] H. Aggarwal, H. D. Arora & V. Kumar (2019). A decision-making problem as an applications of intuitionistic fuzzy set. *International Journal of Engineering and Advanced Technology*, *9*(2), 5259–5261. https://doi.org/10.35940/ijeat.A1053.129219.

[2] M. Akram, G. Shahzadi & X. Peng (2021). Extension of einstein geometric operators to multi-attribute decision making under q-rung orthopair fuzzy information. *Granular Computing*, *6*, 779–795. https://doi.org/10.1007/s41066-020-00233-3.

[3] M. Akram & S. Shumaiza (2021). Multi-criteria decision making based on q-rung orthopair fuzzy promethee approach. *Iranian Journal of Fuzzy Systems*, *18*(5), 107–127. https://doi.org/10.22111/IJFS.2021.6258.

[4] M. Akram & M. Sitara (2021). Decision-making with q-rung orthopair fuzzy graph structures. *Granular Computing*, *7*, 505–526. https://doi.org/10.1007/s41066-021-00281-3.

[5] Z. Ali, T. Mahmood, H. Karamti, K. Ullah, L. Zedam, D. Pamucar & M. Ahmadi (2023). Correction to: Investigation of the brain carcinoma based on generalized variation coefficient similarity measures using complex q-rung orthopair fuzzy information. *Soft Computing*, pp. 1–30. https://doi.org/10.1007/s00500-023-08777-7.

[6] N. Alkan & C. Kahraman (2022). Circular intuitionistic fuzzy topsis method: Pandemic hospital location selection. *Journal of Intelligent and Fuzzy Systems*, *42*(1), 295–316. https://doi.org/10.3233/JIFS-219193.

[7] K. T. Atanassov (1993). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*(1), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[8] K. T. Atanassov (1993). A second type of intuitionistic fuzzy sets. *BUSEFAL*, *56*, 66–70.

[9] K. T. Atanassov (2020). Circular intuitionistic fuzzy sets. *Journal of Intelligent and Fuzzy Systems*, *39*(5), 5981–5986. https://doi.org/10.3233/JIFS-189072.

[10] K. T. Atanassov & G. Gargov (1989). Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *31*(3), 343–349. https://doi.org/10.1016/0165-0114(89)90205-4.

[11] K. T. Atanassov & E. Marinov (2021). Four distances for circular intuitionistic fuzzy sets. *Mathematics*, *9*(10), 1–8. https://doi.org/10.3390/math9101121.

[12] E. Cakir & M. A. Tas (2023). Circular intuitionistic fuzzy decision making and its application. *Expert Systems with Applications*, *225*(1), 1–13. https://doi.org/10.1016/j.eswa.2023.120076.

[13] S. Cali & S. Y. Balaman (2019). A novel outranking based multi criteria group decision making methodology integrating electre and vikor under intuitionistic fuzzy environment. *Expert Systems with Applications*, *119*, 36–50. https://doi.org/10.1016/j.eswa.2018.10.039.

[14] T. Y. Chen (2023). Evolved distance measures for circular intuitionistic fuzzy sets and their exploitation in the technique for order preference by similarity to ideal solutions. *Artificial Intelligence Review*, *56*, 7347–7401. https://doi.org/10.1007/s10462-022-10318-x.

[15] D. Ciucci (2011). Orthopairs: A simple and widely used way to model uncertainty. *Fundamenta Informaticae*, *108*(3-4), 287–304. https://doi.org/10.3233/FI-2011-424.

[16] B. Farhadinia, S. Effati & F. Chiclana (2021). A family of similarity measures for q-rung orthopair fuzzy sets and their applications to multiple criteria decision making. *International Journal of Intelligent Systems*, *36*(4), 1535–1559. https://doi.org/10.1002/int.22351.

[17] H. L. Huang & Y. Guo (2019). An improved correlation coefficient of intuitionistic fuzzy sets. *Journal of Intelligent Systems*, *28*(2), 231–243. http://dx.doi.org/10.1515/jisys-2017-0094.

[18] C. Kahraman & N. Alkan (2021). Circular intuitionistic fuzzy topsis method with vague membership functions: Supplier selection application context. *Notes on Intuitionistic Fuzzy Sets*, *27*(1), 24–52. https://doi.org/10.7546/nifs.2021.27.1.24-52.

[19] C. Kahraman & I. Otay (2021). Extension of vikor method using circular intuitionistic fuzzy sets. In *Intelligent and Fuzzy Techniques for Emerging Conditions and Digital Transformation*, pp. 48–57. Springer Cham, Switzerland. https://doi.org/10.1007/978-3-030-85577-2_6.

[20] M. J. Khan, W. Kumam & N. A. Alreshidi (2022). Divergence measures for circular intuitionistic fuzzy sets and their applications. *Engineering Applications of Artificial Intelligence*, *116*, 105455. https://doi.org/10.1016/j.engappai.2022.105455.

[21] P. Liu, T. Mahmood & Z. Ali (2020). Complex q-rung orthopair fuzzy aggregation operators and their application in multi-attribute group decision making. *Information*, *11*(1), 1–27. https://doi.org/10.3390/info11010005.

[22] W. R. W. Mohd, L. Abdullah, B. Yusoff, C. M. I. C. Taib & J. M. Merigo (2019). An integrated mcdm model based on pythagorean fuzzy sets for green supplier development program. *Malaysian Journal of Mathematical Sciences*, *13*, 23–37.

[23] I. Otay & C. Kahraman (2021). A novel circular intuitionistic fuzzy ahp and vikor methodology: An application to a multi-expert supplier evaluation problem. *Pamukkale University Journal of Engineering Sciences*, *28*(1), 194–207. https://doi.org/10.5505/pajes.2021.90023.

[24] T. Senapati & R. R. Yager (2020). Fermatean fuzzy sets. *Journal of Ambient Intelligence and Humanized Computing*, *11*(2), 663–674. https://doi.org/10.1007/S12652-019-01377-0.

[25] P. Vassilev, R. Parvathi & K. T. Atanassov (2008). Note on intuitionistic fuzzy sets of p-th type. *Issues Intuitionistic Fuzzy Sets Generalized Nets*, *6*, 3–50.

[26] Z. Xu & R. R. Yager (2006). Some geometric aggregation operators based on intuitionistic fuzzy sets. *International Journal of General Systems*, *35*(4), 417–433. https://doi.org/10.1080/03081070600574353.

[27] R. R. Yager (1988). On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Transactions on Systems, Man, and Cybernetics*, *18*, 183–190. https://doi.org/10.1109/21.87068.

[28] R. R. Yager (2013). Pythagorean fuzzy subsets. In *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, pp. 48–57. IEEE, Edmonton, AB, Canada. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375.

[29] R. R. Yager (2017). Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems*, *25*(5), 1222–1230. https://doi.org/10.1109/TFUZZ.2016.2604005.

[30] L. A. Zadeh (1965). Fuzzy sets. *Information And Control*, *8*(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[31] M. I. E. Zulkifly, A. F. Wahab & B. Yusoff (2019). Approximation of intuitionistic fuzzy bezier curve model. *Malaysian Journal of Mathematical Sciences*, *13*, 97–111.

6 Appendix

Table 2: List of abbreviations.

Abbreviations	Explanations
FS	Fuzzy Sets
IFS	Intuitionistic Fuzzy Set
PFS	Pythagorean Fuzzy Set
FFS	Fermatean Fuzzy Set
q-ROFS	q-Rung Orthopair Fuzzy Set
IVIFS	Interval-Valued Intuitionistic Fuzzy Set
CIFS	Circular Intuitionistic Fuzzy Set
Cq-ROFS	Circular q-Rung Orthopair Fuzzy Set
IFIT	Intuitionistic Fuzzy Interpretation Triangle
TOPSIS	Technique for Order of Preference by Similarity to Ideal Solution
VIKOR	VIekriterijumsko KOmpromisno Rangiranje (or Multicriteria Optimization and Compromise Solution)
AHP	Analytic Hierarchy Process
PROMETHEE	Preference Ranking Organization Method for Enrichment Evaluation
MCDM	Multiple Criteria Decision Making
OWA	Ordered Weighted Averaging